Exact solutions of the generalized nonlinear Schrödinger equation with distributed coefficients

V. I. Kruglov, A. C. Peacock, and J. D. Harvey
Physics Department, The University of Auckland, Private Bag 92019, Auckland, New Zealand
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A broad class of exact self-similar solutions to the nonlinear Schrödinger equation (NLSE) with distributed dispersion, nonlinearity, and gain or loss has been found describing both periodic and solitary waves. Appropriate solitary wave solutions applying to propagation in optical fibers and optical fiber amplifiers with these distributed parameters have also been studied in detail. These solutions exist for physically realistic dispersion and nonlinearity profiles. They correspond either to compressing or spreading solitary pulses which maintain a linear chirp or to chirped oscillatory solutions. The stability of these solutions has been confirmed by numerical simulations of the NLSE with perturbed initial conditions. These self-similar propagation regimes are expected to find practical application in both optical fiber amplifier systems and in fiber compressors.

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I. INTRODUCTION

Studies of self-similar solutions of the relevant nonlinear differential equations have been of great value in understanding widely different nonlinear physical phenomena [1]. Although self-similar solutions have been extensively studied in fields such as hydrodynamics and quantum field theory, their application in optics has not been widespread. Some important results have, however, been obtained, with previous theoretical studies considering self-similar behavior in radial pattern formation [2], the self-similar regime of collapse for spiral laser beams in nonlinear media [3], stimulated Raman scattering [4], the evolution of self-written waveguides [5], the formation of Cantor set fractals in soliton systems [6], the nonlinear propagation of pulses with parabolic intensity profiles in optical fibers with normal dispersion [7], and nonlinear compression of chirped solitary waves [8,9].

In this paper, we present the discovery of a broad class of exact self-similar solutions to the nonlinear Schrödinger equation with gain or loss (the generalized NLSE) where all parameters are functions of the distance variable. This class also encloses the set of solitary wave solutions which describes, for example, such physically important applications as the amplification and compression of pulses in optical fiber amplifiers [10]. These linearly chirped solitary wave solutions apply in the anomalous dispersion regime (provided the nonlinearity coefficient is positive), and may be contrasted with the asymptotic solutions appropriate in the normal dispersion regime [11,12]. The importance of the results reported here is twofold. First, the approach leads to a broad class of exact solutions to the nonlinear differential equation in a systematic and transparent way. Some of these solutions have been obtained in the past, either serendipitously [8] or more recently by means of an extension to the widely used inverse-scattering technique which applies to the simple NLSE [15,16], but we emphasize the importance of the use of self-similarity techniques which are broadly applicable for finding solutions to a range of nonlinear partial differential equations, having applications in a variety of other physical situations. The second and more specific significance of these results lies in their potential application to the design of fiber optic amplifiers, optical pulse compressors, and solitary wave based communications links.

II. THE GENERAL CLASS OF SELF-SIMILAR AUTONOMOUS SOLUTIONS

The main feature of our treatment of the generalized NLSE with distributed coefficients is the separation of solutions into definite classes and then the search for a full family of solutions belonging to the appropriate defined class. In this section, we search for solutions which are self-similar and belong to the autonomous class, which yields the quadratic phase with respect to the variable τ. We give appropriate definitions for self-similar solutions and the autonomous class in this section and in more detail in Appendixes A and B. The nonlinear Schrödinger equation with gain in the form used in nonlinear fiber optics is given by

$$i\psi_z = \frac{\beta(z)}{2}\psi_{\tau\tau} - \gamma(z)|\psi|^2\psi + i\frac{g(z)}{2}\psi, \tag{1}$$

where we suppose that all parameters β, γ, and g are the functions of the propagation distance z. This equation describes the amplification or attenuation [when $g(z)$ is negative] of pulses propagating nonlinearly in a single-mode optical fiber where $\psi(z,\tau)$ is the complex envelope of the electrical field in a comoving frame, τ is the retarded time, $\beta(z)$ is the group velocity dispersion (GVD) parameter, $\gamma(z)$ is the nonlinearity parameter, and $g(z)$ is the distributed gain function.

The complex function $\psi(z,\tau)$ can be written as

$$\psi(z,\tau) = U(z,\tau)\exp[i\Phi(z,\tau)], \tag{2}$$

where U and Φ are real functions of z and τ. Using this ansatz, we find the system of two equations for phase $\Phi(z,\tau)$ and amplitude $U(z,\tau)$,

$$U\Phi_z = \frac{\beta(z)}{2}(U\Phi_\tau^2 - U_{\tau\tau}) + \gamma(z)U^3, \tag{3}$$

$$U_z = \frac{\beta(z)}{2}(U\Phi_{\tau\tau} + 2U_\tau\Phi_\tau) + \frac{g(z)}{2}U. \quad (4)$$

In Appendix A, it is shown that in the general case when the coefficients of the NLSE with gain are the functions of the distance z, the amplitude $U(z,\tau)$ of the self-similar solutions has the form

$$U(z,\tau) = \frac{1}{\sqrt{\Gamma(z)}}F(T)\exp\left(\frac{1}{2}G(z)\right), \quad (5)$$

where the scaling variable T and the function $G(z)$ are

$$T = \frac{\tau - \tau_c}{\Gamma(z)}, \quad G(z) = \int_0^z g(z')\mathrm{d}z'. \quad (6)$$

Here $\Gamma(z)$ and $F(T)$ are some functions which we seek, where without loss of generality we can assume that $\Gamma(0)=1$.

In Appendix B, we introduce the class of "autonomous" solutions with the polynomial form of phase,

$$\Phi(z,\tau) = \sum_{n=0}^{N} \phi_n(z)\tau^n. \quad (7)$$

This class of "autonomous" solutions has a special form of the functions $\phi_n(z)$ providing the reduction of Eq. (3) to the set of equations which do not have the explicit dependence on the variable τ. To demonstrate this "autonomous" principle, let us consider the particular case when the phase has the quadratic form

$$\Phi(z,\tau) = a(z) + c(z)(\tau - \tau_c)^2, \quad (8)$$

where τ_c is an arbitrary real constant. Thus we consider the class of self-similar solutions with the phase given by Eq. (8). Then Eq. (3) with the phase (8) can be written

$$U\left(\frac{\mathrm{d}a}{\mathrm{d}z} + \frac{\mathrm{d}c}{\mathrm{d}z}(\tau - \tau_c)^2\right) = 2\beta U c^2(\tau - \tau_c)^2 - \frac{\beta}{2}U_{\tau\tau} + \gamma U^3. \quad (9)$$

This equation contains an explicit dependence on the variable $(\tau-\tau_c)$ which disappears when the terms at monomial $(\tau-\tau_c)^2$ are equals, hence we find the pair of equations

$$\frac{\mathrm{d}c(z)}{\mathrm{d}z} = 2\beta(z)c(z)^2, \quad (10)$$

$$U\frac{\mathrm{d}a(z)}{\mathrm{d}z} = -\frac{\beta(z)}{2}U_{\tau\tau} + \gamma(z)U^3. \quad (11)$$

In Appendix B, it is also proved that the "autonomous" principle yields $N=2$ in Eq. (7), which is actually equivalent to the phase in the form (8). Thus the pair of Eqs. (10) and (11) follows from the "autonomous" principle in the general case when the phase is given by Eq. (7). We note that the "autonomous" principle is not equivalent to the quadratic phase requirement (see Appendix B) since the phase in the form (8) does not yield the system of Eqs. (10) and (11).

Combining Eqs. (4) and (8), we find a third equation for our general class of self-similar autonomous solutions with quadratic phase,

$$U_z = \beta(z)c(z)U + 2\beta(z)c(z)(\tau - \tau_c)U_\tau + \frac{g(z)}{2}U. \quad (12)$$

Taking into account Eq. (5), we find that Eq. (12) will be satisfied if and only if the function $\Gamma(z)$ is defined as

$$\frac{1}{\Gamma(z)}\frac{\mathrm{d}\Gamma(z)}{\mathrm{d}z} = -2\beta(z)c(z). \quad (13)$$

The solutions of Eqs. (10) and (13) are

$$c(z) = \frac{c_0}{1 - c_0 D(z)}, \quad (14)$$

$$\Gamma(z) = 1 - c_0 D(z), \quad (15)$$

where $c_0=c(0)\neq 0$ because the phase should be a quadratic function of variable $(\tau-\tau_c)$ and the function $D(z)$ is

$$D(z) = 2\int_0^z \beta(z')\mathrm{d}z'. \quad (16)$$

Taking into account Eqs. (5) and (11), we find

$$\frac{\mathrm{d}^2F}{\mathrm{d}T^2} + \frac{2\Gamma^2}{\beta}\frac{\mathrm{d}a}{\mathrm{d}z}F - \frac{2\gamma\Gamma}{\beta}\exp[G(z)]F^3 = 0. \quad (17)$$

In the general case, the coefficients in Eq. (17) are the functions of variable z but the function $F(T)$ depends only on the scaling variable T, hence this equation has nontrivial solutions $[F(T)\neq 0]$ if and only if the coefficients in Eq. (17) are constants,

$$-\frac{2\Gamma(z)^2}{\beta(z)}\frac{\mathrm{d}a}{\mathrm{d}z} = \lambda, \quad (18)$$

$$\frac{\gamma(z)\Gamma(z)}{\beta(z)}\exp[G(z)] = \alpha. \quad (19)$$

Here $\lambda=$const, $\alpha=$const, hence Eqs. (18) and (19) yield

$$\lambda = -\frac{2}{\beta(0)}\frac{\mathrm{d}a}{\mathrm{d}z}\bigg|_{z=0}, \quad \alpha = \frac{\gamma(0)}{\beta(0)}, \quad (20)$$

because $\Gamma(0)=1$ and $G(0)=0$. Thus, in the nontrivial case Eq. (17) can be written as

$$\frac{\mathrm{d}^2F}{\mathrm{d}T^2} - \lambda F - 2\alpha F^3 = 0. \quad (21)$$

The solution of Eq. (18) is

$$a(z) = a_0 - \frac{\lambda}{2}\int_0^z \frac{\beta(z')\mathrm{d}z'}{[1 - c_0 D(z')]^2}, \quad (22)$$

where we have used an explicit form for the function $\Gamma(z)$ given by Eq. (15). Here a_0 is an integration constant. We can, however, calculate the integral in this equation to yield the function $a(z)$,

$$a(z) = a_0 - \frac{\lambda D(z)}{4[1 - c_0 D(z)]}. \quad (23)$$

Hence combining Eqs. (8), (14), and (23) we can represent the phase for our solutions in the explicit form

$$\Phi(z,\tau) = a_0 - \frac{\lambda D(z)}{4[1 - c_0 D(z)]} + \frac{c_0(\tau - \tau_c)^2}{1 - c_0 D(z)}. \quad (24)$$

It is useful to write Eq. (19) in the form

$$\rho(z) = \rho(0)[1 - c_0 D(z)]\exp[G(z)], \quad (25)$$

where we define the function $\rho(z)$ as

$$\rho(z) \equiv \frac{\beta(z)}{\gamma(z)}, \quad \rho(0) = \frac{\beta(0)}{\gamma(0)} = \frac{1}{\alpha}. \quad (26)$$

One may differentiate this equation and find

$$g(z) = \frac{1}{\rho(z)}\frac{d\rho(z)}{dz} + \frac{2c_0\beta(z)}{1 - c_0 D(z)}. \quad (27)$$

Evidently, Eqs. (25) and (27) are equivalent and they give the condition for the functions $\beta(z)$, $\gamma(z)$, and $g(z)$, which is the necessary and sufficient condition for the existence of the self-similar solutions of the generalized NLSE (1) with distributed coefficients. These self-similar solutions have the amplitude in the form

$$U(z,\tau) = \frac{1}{\sqrt{1 - c_0 D(z)}} F\left(\frac{\tau - \tau_c}{1 - c_0 D(z)}\right)\exp\left(\frac{1}{2}G(z)\right), \quad (28)$$

where the function $F(T)$ is defined by Eq. (21) and the phase $\Phi(z,\tau)$ is given by Eq. (24). Using these results, we find in Sec. III the set of exact bounded self-similar solutions of Eq. (1).

III. EXACT BOUNDED SELF-SIMILAR SOLUTIONS OF THE GENERALIZED NLSE WITH DISTRIBUTED COEFFICIENTS

In Sec. II it was shown that the phase and the amplitude of self-similar solutions of the generalized Schrödinger equation with distributed coefficients are given by expressions (24) and (28), where the function $F(T)$ is the solution of Eq. (21). Integrating Eq. (21), we find the first-order differential equation,

$$\left(\frac{dF}{dT}\right)^2 = \mu + \lambda F^2 + \alpha F^4, \quad (29)$$

which is integrable in explicit form. Here μ is the integration constant. We also introduce the function $\tilde{F}(u) = F(T)$, where the new variable u is

$$u = \frac{T}{\tau_0} = \frac{\tau - \tau_c}{\tau_0[1 - c_0 D(z)]}. \quad (30)$$

Then Eq. (29) will reduce to the form

$$\left(\frac{d\tilde{F}}{du}\right)^2 = \mu\tau_0^2 + \lambda\tau_0^2\tilde{F}^2 + \alpha\tau_0^2\tilde{F}^4. \quad (31)$$

Using Eqs. (25) and (28), one may find the amplitude of self-similar solutions as

$$U(z,\tau) = \frac{\sqrt{|\alpha\rho(z)|}}{1 - c_0 D(z)}\tilde{F}(u). \quad (32)$$

We suppose below that $\operatorname{sgn}\beta(z) = \operatorname{sgn}\beta(0)$ and $\operatorname{sgn}\gamma(z) = \operatorname{sgn}\gamma(0)$ for $0 \leqslant z < z_0$, hence $\alpha\rho(z) > 0$. Integrating Eq. (31) for the case $\lambda = \tau_0^{-2}$ and $\mu = 0$ when $\beta(z)\gamma(z) < 0$ ($\alpha < 0$) and using Eq. (32), we find the amplitude of the solitary wave solution,

$$U(z,\tau) = \frac{\sqrt{|\rho(z)|}}{\tau_0[1 - c_0 D(z)]}\operatorname{sech}\left(\frac{\tau - \tau_c}{\tau_0[1 - c_0 D(z)]}\right), \quad (33)$$

where τ_0 is the initial pulse width, τ_c is the center of the pulse position, and c_0 is the chirp parameter. These three parameters are arbitrary. Another so-called kink solution follows from Eqs. (31) and (32) to the conditions $\lambda = -2\tau_0^{-2}$ and $\mu = \alpha^{-1}\tau_0^{-4}$ when $\beta(z)\gamma(z) > 0$ ($\alpha > 0$) and yields the amplitude in the form

$$U(z,\tau) = \frac{\sqrt{|\rho(z)|}}{\tau_0[1 - c_0 D(z)]}\tanh\left(\frac{\tau - \tau_c}{\tau_0[1 - c_0 D(z)]}\right), \quad (34)$$

where τ_0, τ_c, and c_0 are also arbitrary parameters. The homogeneous solution (under variable τ) follows in the case $\lambda = -2q^2 \operatorname{sgn}\alpha$ and $\mu = q^4\alpha^{-1}$ for either sign of the product $\beta(z)\gamma(z)$ or α and is

$$U(z,\tau) = \frac{q\sqrt{|\rho(z)|}}{1 - c_0 D(z)}, \quad (35)$$

where q and c_0 are the arbitrary real parameters.

A set of six bounded periodic solutions, depending on four arbitrary real parameters ($\tau_0 > 0$, τ_c, $0 < k < 1$, and c_0) also follows from Eqs. (31) and (32). We use here the standard notations for Jacobian elliptic functions. In Appendix C, the equations for bounded Jacobian elliptic functions are presented in the form as Eq. (31), which yield the following exact periodic solutions [10].

Case 1 [$\alpha > 0$, $\mu = k^2\alpha^{-1}\tau_0^{-4}$, and $\lambda = -(1 + k^2)\tau_0^{-2}$]. Periodic solution when $\beta(z)\gamma(z) > 0$ is

$$U(z,\tau) = \frac{k\sqrt{|\rho(z)|}}{\tau_0[1 - c_0 D(z)]}\operatorname{sn}\left(\frac{\tau - \tau_c}{\tau_0[1 - c_0 D(z)]}, k\right), \quad (36)$$

where $0 < k < 1$ is a free parameter. As $k \to 0$, the function $\operatorname{sn}(u,k) \to \sin u$, hence Eq. (36) also yields the exact asymptotic solution

$$U(z,\tau) = \frac{k\sqrt{|\rho(z)|}}{\tau_0[1 - c_0 D(z)]}\sin\left(\frac{\tau - \tau_c}{\tau_0[1 - c_0 D(z)]}\right), \quad (37)$$

where $k \ll 1$ is a free parameter.

Case 2 [$\alpha < 0$, $\mu = -k^2(1 - k^2)\alpha^{-1}\tau_0^{-4}$, and $\lambda = (2k^2 - 1)\tau_0^{-2}$]. Periodic solution when $\beta(z)\gamma(z) < 0$ is

$$U(z,\tau)=\frac{\mathrm{k}\sqrt{|\rho(z)|}}{\tau_0[1-c_0D(z)]}\mathrm{cn}\left(\frac{\tau-\tau_c}{\tau_0[1-c_0D(z)]},\mathrm{k}\right), \quad (38)$$

where $0<\mathrm{k}<1$ is a free parameter. As $\mathrm{k}\to 0$, the function $\mathrm{cn}(u,\mathrm{k})\to\cos u$, hence Eq. (38) also yields the exact asymptotical solution,

$$U(z,\tau)=\frac{\mathrm{k}\sqrt{|\rho(z)|}}{\tau_0[1-c_0D(z)]}\cos\left(\frac{\tau-\tau_c}{\tau_0[1-c_0D(z)]}\right), \quad (39)$$

where $\mathrm{k}\ll 1$ is a free parameter.

Case 3 $[\alpha<0,\ \mu=(1-\mathrm{k}^2)\alpha^{-1}\tau_0^{-4}$, and $\lambda=(2-\mathrm{k}^2)\tau_0^{-2}]$. Periodic solution when $\beta(z)\gamma(z)<0$ is

$$U(z,\tau)=\frac{\sqrt{|\rho(z)|}}{\tau_0[1-c_0D(z)]}\mathrm{dn}\left(\frac{\tau-\tau_c}{\tau_0[1-c_0D(z)]},\mathrm{k}\right), \quad (40)$$

where $0<\mathrm{k}<1$ is a free parameter. As $\mathrm{k}\to 0$, the function $\mathrm{dn}(u,\mathrm{k})\to 1-\frac{1}{2}\mathrm{k}^2\sin^2 u$, hence Eq. (40) also yields the exact asymptotic solution,

$$U(z,\tau)=\frac{\sqrt{|\rho(z)|}}{\tau_0[1-c_0D(z)]}\times\left[\left(1-\frac{\mathrm{k}^2}{4}\right)+\frac{\mathrm{k}^2}{4}\cos\left(\frac{2(\tau-\tau_c)}{\tau_0[1-c_0D(z)]}\right)\right], \quad (41)$$

where $\mathrm{k}\ll 1$ is a free parameter.

Case 4 $[\alpha<0,\ \mu=-\mathrm{k}^2(1-\mathrm{k}^2)\alpha^{-1}\tau_0^{-4}$, and $\lambda=(2\mathrm{k}^2-1)\tau_0^{-2}]$. Periodic solution when $\beta(z)\gamma(z)<0$ is

$$U(z,\tau)=\frac{\mathrm{k}\sqrt{1-\mathrm{k}^2}\sqrt{|\rho(z)|}}{\tau_0[1-c_0D(z)]}\mathrm{sd}\left(\frac{\tau-\tau_c}{\tau_0[1-c_0D(z)]},\mathrm{k}\right), \quad (42)$$

where $0<\mathrm{k}<1$ is a free parameter. As $\mathrm{k}\to 0$, the function $\mathrm{sd}(u,\mathrm{k})\to\sin u$, hence Eq. (42) also yields the exact asymptotical solution,

$$U(z,\tau)=\frac{\mathrm{k}\sqrt{1-\mathrm{k}^2}\sqrt{|\rho(z)|}}{\tau_0[1-c_0D(z)]}\sin\left(\frac{\tau-\tau_c}{\tau_0[1-c_0D(z)]}\right), \quad (43)$$

where $\mathrm{k}\ll 1$ is a free parameter.

Case 5 $[\alpha>0,\ \mu=\mathrm{k}^2\alpha^{-1}\tau_0^{-4}$, and $\lambda=-(1+\mathrm{k}^2)\tau_0^{-2}]$. Periodic solution when $\beta(z)\gamma(z)>0$ is

$$U(z,\tau)=\frac{\mathrm{k}\sqrt{|\rho(z)|}}{\tau_0[1-c_0D(z)]}\mathrm{cd}\left(\frac{\tau-\tau_c}{\tau_0[1-c_0D(z)]},\mathrm{k}\right), \quad (44)$$

where $0<\mathrm{k}<1$ is a free parameter. As $\mathrm{k}\to 0$, the function $\mathrm{cd}(u,\mathrm{k})\to\cos u$, hence Eq. (44) also yields the exact asymptotic solution,

$$U(z,\tau)=\frac{\mathrm{k}\sqrt{|\rho(z)|}}{\tau_0[1-c_0D(z)]}\cos\left(\frac{\tau-\tau_c}{\tau_0[1-c_0D(z)]}\right), \quad (45)$$

where $\mathrm{k}\ll 1$ is a free parameter.

Case 6 $[\alpha<0,\ \mu=(1-\mathrm{k}^2)\alpha^{-1}\tau_0^{-4}$, and $\lambda=(2-\mathrm{k}^2)\tau_0^{-2}]$. Periodic solution when $\beta(z)\gamma(z)<0$ is

$$U(z,\tau)=\frac{\sqrt{1-\mathrm{k}^2}\sqrt{|\rho(z)|}}{\tau_0[1-c_0D(z)]}\mathrm{nd}\left(\frac{\tau-\tau_c}{\tau_0[1-c_0D(z)]},\mathrm{k}\right), \quad (46)$$

where $0<\mathrm{k}<1$ is a free parameter. As $\mathrm{k}\to 0$, the function $\mathrm{nd}(u,\mathrm{k})\to(1-\frac{1}{2}\mathrm{k}^2\sin^2 u)^{-1}$, hence Eq. (46) also yields the exact asymptotic solution,

$$U(z,\tau)=\frac{\sqrt{1-\mathrm{k}^2}\sqrt{|\rho(z)|}}{\tau_0[1-c_0D(z)]}\times\left[\left(1+\frac{\mathrm{k}^2}{4}\right)-\frac{\mathrm{k}^2}{4}\cos\left(\frac{2(\tau-\tau_c)}{\tau_0[1-c_0D(z)]}\right)\right], \quad (47)$$

where $\mathrm{k}\ll 1$ is a free parameter.

We note that six unbounded periodic solutions have the same form as bounded periodic solutions with $\lambda=\nu(\mathrm{k})\tau_0^{-2}$ and $\mu=\eta(\mathrm{k})\alpha^{-1}\tau_0^{-4}$,

$$U(z,\tau)=\frac{\mathrm{p}(\mathrm{k})\sqrt{|\rho(z)|}}{\tau_0[1-c_0D(z)]}\mathrm{f}\left(\frac{\tau-\tau_c}{\tau_0[1-c_0D(z)]},\mathrm{k}\right), \quad (48)$$

where the function $\mathrm{f}(u,\mathrm{k})$ is given by Jacobian elliptic functions as $\mathrm{ns}(u,\mathrm{k})$, $\mathrm{nc}(u,\mathrm{k})$, $\mathrm{ds}(u,\mathrm{k})$, $\mathrm{dc}(u,\mathrm{k})$, $\mathrm{sc}(u,\mathrm{k})$, and $\mathrm{cs}(u,\mathrm{k})$. Here the constants $\mathrm{p}(\mathrm{k})$, $\nu(\mathrm{k})$, and $\eta(\mathrm{k})$ depending on k are also defined by Eq. (31) and one may find them.

We consider the generalization of the solutions found above using some transformation to traveling solutions $\tilde{\psi}^{(s)}(z,\tau)$ which also satisfy the generalized nonlinear Schrödinger equation (1), where index s indicates some definite solution derived in this section. We seek these traveling solutions in the form

$$\tilde{\psi}^{(s)}(z,\tau)=\psi^{(s)}(z,\tau')\exp[i\phi(z,\tau)], \quad (49)$$

where the new variable τ' is

$$\tau'=\tau-\int_0^z \mathrm{v}(z')\mathrm{d}z'. \quad (50)$$

Here $\mathrm{v}(z)$ is some real function of z, $\psi^{(s)}(z,\tau)$ denote some particular solution found in this section, and the phase $\phi(z,\tau)$ is an unknown real function. We note that Eq. (50) yields

$$\frac{\partial(\tau-\tau')}{\partial z}=\mathrm{v}(z),$$

hence, if we consider variable z in Eq. (1) as the time and variable τ as the propagating distance, then the function $\mathrm{v}(z)$ is the velocity of generalized traveling wave solution $\tilde{\psi}^{(s)}(z,\tau)$.

Substitution of the traveling wave solution (49) in Eq. (1) yields the equation

$$i[\beta(z)\phi_\tau+\mathrm{v}(z)]\psi^{(s)}_{\tau'}+\left(\phi_z-\frac{\beta(z)}{2}\phi_\tau^2\right)\psi^{(s)}+i\frac{\beta(z)}{2}\phi_{\tau\tau}\psi^{(s)}=0, \quad (51)$$

where $\psi^{(s)}=\psi^{(s)}(z,\tau')$ and $\phi=\phi(z,\tau)$. We note that in deriving this equation, we take into account that the function $\psi^{(s)}(z,\tau)$ also satisfies Eq. (1).

Equation (51) can be satisfied if we suppose that the first and second terms in this equation are identically equal to zero, which means

$$\phi_\tau = -\frac{\mathrm{v}(z)}{\beta(z)}, \quad \phi_z = \frac{\beta(z)}{2}\phi_\tau^2 = \frac{\mathrm{v}^2(z)}{2\beta(z)}. \tag{52}$$

It is clear from the first equation of the system (52) that $\phi_{\tau\tau}=0$, hence in this case the last term in Eq. (51) is equal to zero automatically. Integration of the first equation of the system (52) leads to the solution

$$\phi(z,\tau) = -\frac{\mathrm{v}(z)}{\beta(z)}\tau + \chi(z). \tag{53}$$

Then the second equation of the system (52) can be written

$$-\tau\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{v}(z)}{\beta(z)}\right) = \frac{\mathrm{v}^2(z)}{2\beta(z)} - \frac{\mathrm{d}}{\mathrm{d}z}\chi(z), \tag{54}$$

where $\chi(z)$ is an arbitrary function of z. Since the right-hand side of Eq. (54) is a function of z, the left-hand side of this equation must be equal to zero, which yields

$$\mathrm{v}(z) = \omega\beta(z), \tag{55}$$

where ω is an arbitrary real parameter. Combining Eqs. (54) and (55) and integrating, we find the function $\chi(z)$,

$$\chi(z) = \frac{\omega^2}{2}\int_0^z \beta(z')\mathrm{d}z' + \kappa, \tag{56}$$

where κ is an arbitrary real parameter. Hence, using Eqs. (53) and (56), we find the phase $\phi(z,\tau)$ as

$$\phi(z,\tau) = \kappa + \frac{\omega^2}{4}D(z) - \omega\tau. \tag{57}$$

This phase yields the generalized traveling solutions (49) in the form

$$\tilde{\psi}^{(s)}(z,\tau) = \psi^{(s)}\left(z,\tau - \frac{\omega}{2}D(z)\right)\exp\left[i\left(\kappa + \frac{\omega^2}{4}D(z) - \omega\tau\right)\right]. \tag{58}$$

We note that in the particular case $\beta(z)=\beta=$const the transformation (58) reduces to the Galileian transformation,

$$\tilde{\psi}^{(s)}(z,\tau) = \psi^{(s)}(z,\tau - \mathrm{v}z)\exp\left[i\left(\kappa + \frac{\mathrm{v}^2}{2\beta}z - \frac{\mathrm{v}}{\beta}\tau\right)\right].$$

Here $\mathrm{v}=\omega\beta=$const is the velocity [if z is the time and τ is the propagating distance in Eq. (1)].

In conclusion, we note that the transformations (58) form a one-parameter Abelian Lie group. In fact, without loss of generality we can put $\kappa=0$ and write the transformation (58) as $\psi_\omega^{(s)}=\mathbf{T}_\omega\psi_0^{(s)}$, where $\tilde{\psi}^{(s)}\equiv\psi_\omega^{(s)}$ and $\psi^{(s)}\equiv\psi_0^{(s)}$. Then one may find

$$\psi_{\omega_1+\omega_2}^{(s)} = \mathbf{T}_{\omega_2}\mathbf{T}_{\omega_1}\psi_0^{(s)},$$

hence the operators (transformations) $\mathbf{T}_\omega$ are the elements of Abelian Lie group with $\omega\in R_1$: $\mathbf{T}_{\omega_1+\omega_2}=\mathbf{T}_{\omega_2}\mathbf{T}_{\omega_1}=\mathbf{T}_{\omega_1}\mathbf{T}_{\omega_2}$. Thus, applying the transformation (58) to the exact solutions found in this section [with phase given by Eq. (24)], we get the generalized traveling solutions depending on the arbitrary frequency parameter ω.

IV. APPLICATIONS FOR FIBER AMPLIFIER SYSTEMS AND FIBER COMPRESSORS

In Sec. II it was shown that exact self-similar solutions of Eq. (1) take place only when three functions $\beta(z)$, $\gamma(z)$, and $g(z)$ satisfy Eq. (25) or equivalent Eq. (27). Thus we have three different cases.

(i) The differentiable functions $\beta(z)$ and $\gamma(z)$ are given, and then the function $g(z)$ is defined by Eq. (27).

(ii) The functions $\beta(z)$ and $g(z)$ are given, and then the function $\gamma(z)$ follows from Eq. (25),

$$\gamma(z) = \left(\frac{\gamma_0}{\beta_0}\right)\frac{\beta(z)}{1-c_0D(z)}\exp[-G(z)], \tag{59}$$

where $\beta_0=\beta(0)$ and $\gamma_0=\gamma(0)$.

(iii) The functions $\gamma(z)$ and $g(z)$ are given and the function $\beta(z)$ needs to be defined. We solve this problems using the ansatz

$$1-c_0D(z) = \exp[-f(z)], \tag{60}$$

where, as follows from this ansatz, the new function $f(z)$ satisfies the condition $f(0)=0$. We note that the solitary wave solution (33) and ansatz (60) yield the width of the pulse as

$$W(z) \equiv \tau_0[1-c_0D(z)] = \tau_0\exp[-f(z)]. \tag{61}$$

Combining Eqs. (27) and (60), one may find

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}f(z) - \left(g(z) + \frac{1}{\gamma(z)}\frac{\mathrm{d}\gamma(z)}{\mathrm{d}z}\right)\frac{\mathrm{d}}{\mathrm{d}z}f(z) = 0. \tag{62}$$

We also find that ansatz (60) yields the function $\beta(z)$ in the form

$$\beta(z) = \frac{1}{2c_0}\frac{\mathrm{d}f(z)}{\mathrm{d}z}\exp[-f(z)]. \tag{63}$$

Taking into account the condition $f(0)=0$ and Eq. (63), one may find the boundary conditions for Eq. (62),

$$f|_{z=0} = 0, \qquad \left.\frac{\mathrm{d}f}{\mathrm{d}z}\right|_{z=0} = 2c_0\beta_0, \tag{64}$$

hence the solution of Eq. (62) is

$$f(z) = \frac{2c_0\beta_0}{\gamma_0}\int_0^z \gamma(z')\exp[G(z')]\mathrm{d}z'. \tag{65}$$

Combining Eqs. (63) and (65), we find the function $\beta(z)$ in the case (iii) when the functions $\gamma(z)$ and $g(z)$ are given,

$$\beta(z) = \frac{\beta_0\gamma(z)}{\gamma_0}\exp\left(G(z) - \frac{2c_0\beta_0}{\gamma_0}\int_0^z \gamma(z')\exp[G(z')]\mathrm{d}z'\right). \tag{66}$$

Using Eqs. (61) and (65), one may also find in this case the width as

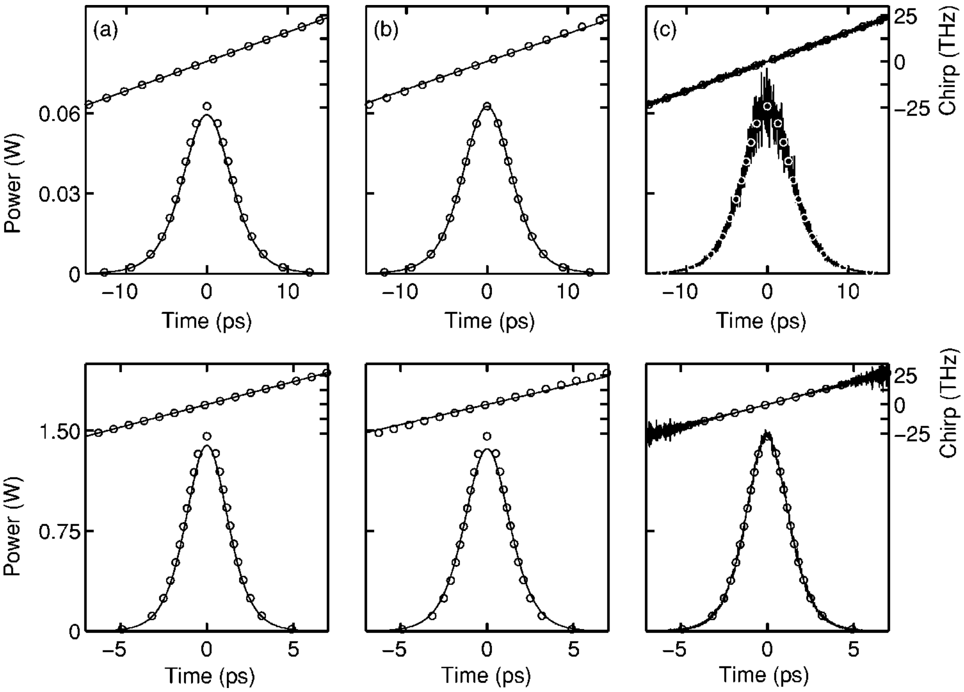

FIG. 1. Stability of the solutions under the influence of perturbations. Upper panel-input pulse, lower panel-output pulse. (a) Input power 95% of ideal power, (b) chirp 95% of ideal chirp, (c) 5% random noise on amplitude and phase. These simulations were for the case of a hyperbolic gain profile with 9 m of fiber which generated 10 dB of gain with constant dispersion and nonlinearity. The circles indicate the ideal input and output pulses (i.e., the exact solution) and the solid lines are the numerically generated simulations.

$$W(z) = \tau_0 \exp\left(-\frac{2c_0\beta_0}{\gamma_0}\int_0^z \gamma(z')\exp[G(z')]\mathrm{d}z'\right). \quad (67)$$

Let us consider some particular cases when $\gamma(z)=\text{const}$. In the case $g(z)=0$, Eqs. (66) and (67) yield desired functions as

$$\beta(z) = \beta_0 \exp(-2c_0\beta_0 z), \quad (68)$$

$$W(z) = \tau_0 \exp(-2c_0\beta_0 z). \quad (69)$$

Let us assume $g(z)=g_0=\text{const}\neq 0$. Then the functions $\beta(z)$ and $W(z)$ are

$$\beta(z) = \beta_0 \exp\left[g_0 z - \frac{2c_0\beta_0}{g_0}[\exp(g_0 z)-1]\right], \quad (70)$$

$$W(z) = \tau_0 \exp\left[-\frac{2c_0\beta_0}{g_0}[\exp(g_0 z)-1]\right]. \quad (71)$$

In the more general case, $g(z)=g_0\exp(\Lambda z)$, Eqs. (66) and (67) yield required functions as

$$\beta(z) = \beta_0 \exp\left[\sigma[\exp(\Lambda z)-1] - \frac{2c_0\beta_0}{\Lambda e^{\sigma}}\int_{\sigma}^{\sigma\exp(\Lambda z)}\frac{\exp(t)}{t}\mathrm{d}t\right], \quad (72)$$

$$W(z) = \tau_0 \exp\left[-\frac{2c_0\beta_0}{\Lambda e^{\sigma}}\int_{\sigma}^{\sigma\exp(\Lambda z)}\frac{\exp(t)}{t}\mathrm{d}t\right], \quad (73)$$

where $\sigma=g_0\Lambda^{-1}$.

V. STABILITY OF THE EXACT SOLUTIONS AND CORRESPONDENCE WITH THE INVERSE SCATTERING METHOD

We have proved numerically the stability of the evolution of these self-similar solutions under initial small perturbations and also under nonideal parameter profiles. Typical results of numerical simulations are shown in Fig. 1. These numerical simulations show that the evolution is more sensitive to the initial chirp than to perturbations of the amplitude, but in both cases the addition of small amounts of random noise to the input pulse amplitude and phase did not significantly affect the evolution. Indeed, in both cases the pulses evolved towards the ideal form, indicating the stability of the solutions. This result is to be expected in light of the inverse-scattering technique, which can be applied to this problem (see below in this section).

Let us consider the influence of small deviations of the solutions from the ideal form when the functions $\beta(z)$, $\gamma(z)$, and $g(z)$ do not satisfy to the condition given by Eq. (27). We suppose that the function $\delta s(z)$ in the equation

$$\delta s(z) = \frac{1}{\rho(z)}\frac{\mathrm{d}\rho(z)}{\mathrm{d}z} + \frac{2c_0\beta(z)}{1-c_0D(z)} - g(z) \quad (74)$$

is not zero. Then integrating this equation, we find

$$\rho(z) = \rho(0)[1-c_0D(z)]\exp[G(z)+\delta S(z)], \quad (75)$$

where

$$\delta S(z) = \int_0^z \delta s(z')\mathrm{d}z'.$$

The requirement of small deviations of the self-similar solutions follows from this equation and can be written as

$$|\exp[\delta S(z)] - 1| \ll 1. \tag{76}$$

It is evident that this equation will be satisfied only when $|\delta S(z)| \ll 1$. Combining Eqs. (75) and (76), we find the desired criterion

$$\left| \frac{\rho(z)}{\rho(0)} \frac{\exp[-G(z)]}{1 - c_0 D(z)} - 1 \right| \ll 1. \tag{77}$$

We note that this simple criterion is very useful for the design of an amplifying pulse compressor, as the self-similar solutions will be compressed while (for such z) this criterion holds.

Let us suppose below that $\beta(z) = \beta_0(z) + \delta\beta(z)$, $\gamma(z) = \gamma_0(z) + \delta\gamma(z)$, and $g(z) = g_0(z) + \delta g(z)$, where the functions $\beta_0(z)$, $\gamma_0(z)$, and $g_0(z)$ satisfy the condition

$$\rho_0(z) = \rho_0(0)[1 - c_0 D_0(z)]\exp[G_0(z)],$$

where $\rho_0(z) = \beta_0(z)/\gamma_0(z)$, $D_0(z) = 2\int_0^z \beta_0(z')dz'$, and $G_0(z) = \int_0^z g_0(z')dz'$. This means that the generalized NLSE with the parameters $\beta_0(z), \gamma_0(z)$, and $g_0(z)$ has self-similar solutions classified in this paper and the functions $\delta\beta(z)$, $\delta\gamma(z)$, and $\delta g(z)$ we will consider as some deviations which take place in the optical compressor. We also suppose that $\delta\beta(0) = 0$, $\delta\gamma(0) = 0$, $|\delta\beta(z)/\beta_0(z)| \ll 1$, and $|\delta\gamma(z)/\gamma_0(z)| \ll 1$. Then Eq. (77) has the form

$$\left| \frac{1 - c_0 D_0(z)}{1 - c_0 D(z)} \left(1 + \frac{\delta\beta(z)}{\beta_0(z)} - \frac{\delta\gamma(z)}{\gamma_0(z)} \right) e^{-\delta G(z)} - 1 \right| \ll 1. \tag{78}$$

Here $D(z) = D_0(z) + \delta D(z)$, $\delta D(z)$, and $\delta G(z)$ are given by equations

$$\delta D(z) = 2\int_0^z \delta\beta(z')dz', \quad \delta G(z) = \int_0^z \delta g(z')dz'.$$

In our case, $c_0 D_0(z) \to 1$ with increasing z because the width in the optical compressor decreases when z is increasing. Hence, if $\delta\beta(z) \neq 0$, the first term in Eq. (78) tends to zero with increasing z and the inequality (78) is broken. Thus to avoid this problem, we may put $\delta\beta(z) = 0$. Then the inequality (78) reduces as

$$\left| \left(1 - \frac{\delta\gamma(z)}{\gamma_0(z)} \right) e^{-\delta G(z)} - 1 \right| \ll 1. \tag{79}$$

This inequality will be satisfied when $|\delta G(z)| \ll 1$, and hence the best approach is to design the optical compressor under the conditions $\beta(z) = \beta_0(z)$ [$\delta\beta(z) = 0$] and

$$\left| \frac{\delta\gamma(z)}{\gamma_0(z)} \right| \ll 1, \quad \left| \int_0^z \delta g(z')dz' \right| \ll 1. \tag{80}$$

The distance $z = z_0$ where these conditions fail is the critical distance for the self-similar solutions, but numerical simulations have shown that the pulse can be compressed for $z > z_0$ with some nonideal form provided the gain increases rapidly.

Finally, we consider the correspondence between the solutions obtained for the generalized NLSE and the standard NLSE which is integrable by inverse-scattering techniques. We define new dimensionless variables t, s and dimensionless function $w(s,t)$ as

$$t = u(z,\tau) \equiv \frac{\tau - \tau_c}{\tau_0[1 - c_0 D(z)]}, \quad s = s(z) \equiv \frac{|D(z)|}{2\tau_0^2[1 - c_0 D(z)]}, \tag{81}$$

$$\psi(z,\tau) = N(z)\exp\left(\frac{ic_0(\tau - \tau_c)^2}{1 - c_0 D(z)} \right) w(s,t), \tag{82}$$

where τ_0, τ_c, and c_0 are the arbitrary real constants and $N(z)$ is an arbitrary real function of z [Im $N(z) = 0$]. We note that the variable $t = u(z,\tau)$ was defined above [see Eq. (30)] and the function $s(z)$ can be written as $s(z) = 1/\mathcal{L}_D(z)$, where the function $\mathcal{L}_D(z)$ is introduced in the paper [17]. The function $\mathcal{L}_D(z)$ is a generalization of the dispersion length in dimensionless form, appropriate for the propagation of chirped pulses (see [17]). The transformation given by Eq. (82) is connected with the main part of the phase function $\Phi(z,\tau)$ [last term in Eq. (24)]. One may find that the transformations Eqs. (81) and (82) reduce the generalized NLSE Eq. (1) to the dimensionless form

$$iw_s = \frac{\beta(z)}{2|\beta(z)|} w_{tt} - \tau_0^2[1 - c_0 D(z)]^2 \frac{\gamma(z)}{|\beta(z)|} \times N(z)^2 |w|^2 w + i\frac{1}{2}\sigma(s)w, \tag{83}$$

where the gain function $\sigma(s)$ is

$$\sigma(s) = \frac{\tau_0^2[1 - c_0 D(z)]^2}{|\beta(z)|} \left(g(z) + \frac{2c_0\beta(z)}{1 - c_0 D(z)} - \frac{2}{N(z)}\frac{dN(z)}{dz} \right). \tag{84}$$

Because $N(z)$ is an arbitrary function, it is helpful to define this function from the condition that the coefficient at the nonlinear term in Eq. (83) is a constant. This yields the function $N(z)$ as

$$N(z) = \frac{\sqrt{|\rho(z)|}}{\tau_0[1 - c_0 D(z)]}, \tag{85}$$

where $\rho(z) = \beta(z)/\gamma(z)$. It is easy to see that this normalizing function $N(z)$ in Eq. (82) is proportional to the normalizing function in Eq. (32) [see also the exact solutions given by Eqs. (33)–(48)]. Hence in this particular case Eq. (83) takes the form

$$iw_s = \frac{\beta(z)}{2|\beta(z)|} w_{tt} - \frac{\gamma(z)}{|\gamma(z)|} |w|^2 w + i\frac{1}{2}\sigma(s)w, \tag{86}$$

where the gain function $\sigma(s)$ is defined as

$$\sigma(s) = \frac{\tau_0^2[1 - c_0 D(z)]^2}{|\beta(z)|} \left(g(z) - \frac{2c_0\beta(z)}{1 - c_0 D(z)} - \frac{1}{\rho(z)}\frac{d\rho(z)}{dz} \right). \tag{87}$$

We assume here that the function $s = s(z)$ is defined by Eq. (81). Note that Eq. (87) can be written also in the form

$$\sigma(s) = \frac{1}{\theta(z(s))}\frac{d\theta(z(s))}{ds}, \tag{88}$$

where the function $\theta(z)$ is

$$\theta(z) = \frac{1 - c_0 D(z)}{\rho(z)} \exp[G(z)], \tag{89}$$

and $z=z(s)$ is the inverse of the function $s=s(z)$ given by Eq. (81). Considering the case when the functions $\beta(z)$ and $\gamma(z)$ do not change the sign on the some interval $z \in (0, z_0)$ and assuming that Eq. (27) holds, we find that Eq. (86) is the standard NLSE with constant coefficients because in this case $\beta/|\beta| = \mathrm{sgn}\beta = \mathrm{const}$, $\gamma/|\gamma| = \mathrm{sgn}\ \gamma = \mathrm{const}$, and $\sigma(s)=0$.

Thus the transformation given by Eqs. (81) and (82) and Eq. (85) yields one-to-one correspondence between the standard NLSE with constant coefficients and generalized NLSE when Eq. (27) holds. Hence all solutions of the standard NLSE have their counterparts in Eq. (1). Because the standard NLSE can be solved exactly by using the inverse-scattering method [13,14], we can find all exact solutions of the generalized NLSE with distributed coefficients when the condition Eq. (27) holds. In particular, one can write all multiple soliton solutions of Eq. (1) when Eq. (27) holds. Finally, we note that the nontrivial transformation found here [Eqs. (81), (82), and (85) subject to Eq. (27)] together with the inverse-scattering method [13,14] generalizes the treatment developed in the above sections to include higher-order solutions analogous to the well known higher-order solitons.

VI. THE OSCILLATORY SOLUTIONS

While the solitary hyperbolic secant shaped pulses have also been discovered using different mathematical techniques [10,15], the oscillatory solutions which are based on elliptic Jacobean functions seem at first sight to be more of a curiosity than an experimentally applicable solution. This is because the solutions are not localized in time and hence the linear chirp involves in principle an unbounded range of frequencies. One important characteristic of these exact solutions, however, is their stability, and this characteristic also applies to a linearly chirped burst of amplitude modulated light. We have analyzed this situation numerically using an amplitude modulated pulse generated by windowing the exact solutions with a range of pulse envelopes. The oscillatory amplitudes can be either close to sinusoidal in shape (which is a limiting case of the elliptic Jacobean functions) or exact elliptic Jacobean functions, and in all cases the amplitude modulated pulse evolves self-similarly in the same way as the solitary wave solutions.

The exact solitary solutions have a hyperbolic secant shape but differ from fundamental soliton solutions by their linear chirp and their continuously increasing amplitude and decreasing pulse width. An example is shown in Fig. 2, where the input and output pulse shapes and chirps are displayed in the two left-hand panels, for the case of hyperbolic gain, with constant dispersion and nonlinearity. Propagation through this amplifying fiber leads to an order of magnitude compression and amplification of the pulse, together with a correspondingly increased frequency chirp. The two remaining panels show the self-similar evolution of a chirped oscillatory pulse generated by windowing the exact oscillatory solution with a super-Gaussian envelope as it passes through the same amplifying fiber. The amplitude modulated pulse and its associated chirp scale in exactly the same way as the hyperbolic secant solution, but the frequency of the amplitude modulation has now increased by an order of magnitude. The modulated pulse simply scales to preserve the relationship between the amplitude oscillations and the width of the envelope. All of these similariton solutions arise as a result of the interplay between dispersion and nonlinearity, and the use of these pulses avoids the deleterious effect of wave breaking which can otherwise disrupt high-power optical pulses in single-mode amplifying fibers. The experimental realization of these new chirped oscillatory solutions could lead to new applications, for example in the generation of tunable THz radiation by electro-optic conversion.

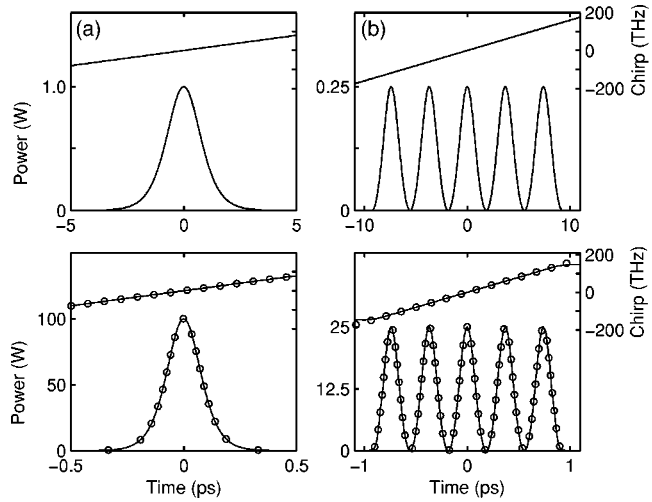

FIG. 2. Compression of a solitary hyperbolic secant chirped pulse (a) and an amplitude modulated super-Gaussian chirped pulse (b) through the same region of distributed gain fiber. Note the order of magnitude compression of both pulses and the corresponding increase in frequency of the modulation for the amplitude modulated pulse. The overall gain in this example is 10 dB. The circles indicate the exact solution and the solid lines are the numerically generated simulations.

VII. DISCUSSION

The techniques used to search for self-similar solutions to nonlinear differential equations have not previously been applied extensively in optics research, but they have been shown here to lead straightforwardly to the development of a broad class of solutions to the generalized nonlinear Schrödinger equation. These include propagating chirped oscillatory solutions, "kink" solutions, and solitary wave solutions. Of these, the solitary wave solutions and the oscillatory solutions are likely to find the most practical applications. The solitary wave solutions have also been found recently using an extension to the inverse scattering technique [15,16], but we wish here to emphasize the self-similar nature of these solitary solutions, which continually compress or expand while propagating with a developing chirp in the presence of gain or loss, respectively. In the case of propagation in the presence of gain, the solitary waves evolve towards a δ function, although naturally, other higher-order terms (which are not taken into account in the NLSE as written here) would preclude such an ultimate fate for an optical pulse in a single-mode fiber. We also note that all appropriate solutions of the generalized NLSE may be found by transformation given by Eqs. (81) and (82) and Eq. (85) together with the standard inverse-scattering method [13,14] and the additional condition Eq. (27).

The chirped solitary wave solutions have been called solitons [15,16] in view of their behavior during collisions, after which they are able to regain their original form. Indeed there has been a tendency recently to use the term "soliton" for all solitary waves in optics. The word soliton was first introduced, however, for a particular kind of solitary wave solution to emphasize that the behavior of these solutions is particlelike, implying that the energy propagates in the form of localized "packets" with constant energy. Since this is clearly not the case in the context of these solutions which describe propagation in the presence of dissipation or gain, we have avoided the use of the term soliton to describe the solitary wave self- similar solutions. In the case of propagation in the normal dispersion regime, where the use of self-similar techniques shows that the asymptotic solitary wave solution is a parabolic pulse [12], we have introduced the term similariton to describe self-similar solitary parabolic solution. Progress has also been made recently in developing similariton lasers utilizing these solitary wave solutions [18,19], and we feel that this term is also more appropriate to describe the linearly chirped self-similar hyperbolic secant pulses in the anomalous dispersion regime, and to reserve the term soliton for constant energy pulses.

The chirped oscillatory solutions have both a chirped underlying carrier wave and a chirped oscillatory envelope, and may find application as a new way to generate very high frequency amplitude modulated light waves. These self-similar solutions are exact solutions for all z (unlike the asymptotically exact parabolic solutions) and may well find new applications in fiber optic amplifiers and compressors, particularly in view of their stability under perturbations which may well lead to reduced noise in generated pulse stream. In general, the full experimental exploitation of these solutions requires optical fibers with tailor-made dispersion profiles and nonlinearity profiles. While this is clearly a technical challenge, such fibers may well become available, enabling the development of new types of pulsed and oscillatory light sources in the future.

APPENDIX A: THE AMPLITUDE FORM OF SELF-SIMILAR SOLUTIONS

In the general case, the amplitude of self-similar solutions depends on the scaling variable T which is a combination of variables $(\tau-\tau_c)$ and some function $\Gamma(z)$ of variable z. However, in the case of the generalized NLSE with distributed coefficients, the amplitude depends on two independent variables which are appropriately chosen as T and z. Evidently, one may choose other variables. Since we search for self-similar solutions, in the general case the amplitude is the product of two functions where one of them depends on variable T and another one is the function of variable z. Really, the self-similar solutions have scaling structure [1], hence we can represent the amplitude $U(z,\tau)$ in the form

$$U(z,\tau) = S(z)F(T), \quad T = \frac{\tau-\tau_c}{\Gamma(z)}, \tag{A1}$$

where without loss of generality we can suppose that

$$S(0) = 1, \quad \Gamma(0) = 1. \tag{A2}$$

It is easy to show that the generalized NLSE has the energy integral of motion [20], which is

$$I(z) = I(0)\exp\left(\int_0^z g(z')dz'\right), \tag{A3}$$

where the function $I(z)$ yields the energy of the pulse at the distance z and it is given by

$$I(z) = \int_{-\infty}^{+\infty} |\psi(z,\tau)|^2 d\tau. \tag{A4}$$

Combining Eqs. (A1) and (A4), we find

$$I(z) = S(z)^2\int_{-\infty}^{+\infty} F\left(\frac{\tau-\tau_c}{\Gamma(z)}\right)^2 d\tau = S(z)^2\Gamma(z)\int_{-\infty}^{+\infty} F(T)^2 dT. \tag{A5}$$

Using Eqs. (A3) and (A5), and taking into account Eq. (A2), we have

$$S(z)^2\Gamma(z) = \exp\left(\int_0^z g(z')dz'\right). \tag{A6}$$

Hence, substitution of the function $S(z)$ from Eq. (A6) to Eq. (A1) yields the form of the amplitude as

$$U(z,\tau) = \frac{1}{\sqrt{\Gamma(z)}}F(T)\exp\left(\frac{1}{2}G(z)\right). \tag{A7}$$

Thus, we have found the general structure of the amplitude of the generalized NLSE with distributed coefficients.

APPENDIX B: THE AUTONOMOUS PRINCIPLE

We define the class of autonomous solutions as the class of solutions which reduce Eq. (3) to the set of equations that have no explicit dependence on the variable τ. At first, as an example, let us consider some generalization of Eq. (8) supposing that the phase has the form

$$\Phi(z,\tau) = a(z) + c(z)(\tau-\tau_c)^n. \tag{B1}$$

Then Eqs. (3) and (B1) yield the equation

$$U\left(\frac{\mathrm{d}a}{\mathrm{d}z}+\frac{\mathrm{d}c}{\mathrm{d}z}(\tau-\tau_c)^n\right)=\frac{\beta}{2}Un^2c^2(\tau-\tau_c)^{2n-2}-\frac{\beta}{2}U_{\tau\tau}+\gamma U^3. \tag{B2}$$

Using our autonomous principle, we find that the exponents at $(\tau-\tau_c)$ in the left- and right-hand sides of Eq. (B2) should be the same, hence $n=2n-2$ or $n=2$. So we find Eqs. (10) and (11) in this more general case too. Let us suppose that in the general case, the phase has the polynomial form given by

$$\Phi(z,\tau)=\sum_{n=0}^{N}\phi_n(z)\tau^n. \tag{B3}$$

Then combining Eqs. (3) and (B3) and using the autonomous principle, one may find the pair of equations

$$U\frac{\mathrm{d}\phi_0}{\mathrm{d}z}=\frac{\beta}{2}U\phi_1^2-\frac{\beta}{2}U_{\tau\tau}+\gamma U^3, \tag{B4}$$

$$\sum_{n=1}^{N}\frac{\mathrm{d}\phi_n}{\mathrm{d}z}\tau^n=\beta\phi_1\sum_{n=2}^{N}n\phi_n\tau^{n-1}+\frac{\beta}{2}\left(\sum_{n=2}^{N}n\phi_n\tau^{n-1}\right)^2. \tag{B5}$$

For an example $N=2$, Eq. (B5) yields

$$\frac{\mathrm{d}\phi_1}{\mathrm{d}z}=2\beta\phi_1\phi_2,\quad \frac{\mathrm{d}\phi_2}{\mathrm{d}z}=2\beta\phi_2^2. \tag{B6}$$

It is easy to find that for $N=3$, Eq. (B5) yields Eqs. (B6) and $\phi_3=0$. Now we can prove that for any integer $N\geqslant 3$, Eq. (B5) leads to Eqs. (B6) and $\phi_n=0$ for $n>2$. Really, we can assume this statement for $N=M>3$. Then for $N=M+1$, Eq. (B5) yields

$$\begin{aligned}\sum_{n=1}^{M}\frac{\mathrm{d}\phi_n}{\mathrm{d}z}\tau^n+\frac{\mathrm{d}\phi_{M+1}}{\mathrm{d}z}\tau^{M+1}&=\beta\phi_1\sum_{n=2}^{M}n\phi_n^{n-1}\tau^{n-1}\\&+\beta\phi_1(M+1)\phi_{M+1}\tau^M+\frac{\beta}{2}\left(\sum_{n=2}^{M}n\phi_n\tau^{n-1}\right)^2\\&+\beta(M+1)\phi_{M+1}\tau^M\sum_{n=2}^{M}n\phi_n\tau^{n-1}+\frac{\beta}{2}(M+1)^2\phi_{M+1}^2\tau^{2M}.\end{aligned} \tag{B7}$$

Evidently, the last term in this equation is equal to zero because no other term in Eq. (B7) is proportional to τ^{2M} for $M>3$, hence $\phi_{M+1}=0$. Due to $\phi_{M+1}=0$, Eq. (B7) reduces to Eq. (B5) for $N=M$, but above we have assumed that for $N=M>3$, Eq. (B5) leads to Eqs. (B6) and $\phi_n=0$ for $n>2$, hence we have proved this statement for any integer $N\geqslant 3$. It is clear that this result is valid also when $N=\infty$.

Moreover, one may perform the full treatment which we have developed in Sec. II using the phase in the form (B3) for $N=2$. Then in this case we have Eqs. (B4) and (B6) instead of Eqs. (10) and (11). Evidently in this case we should also use the equation

$$U_z=\beta(z)\phi_2(z)U+\beta(z)[\phi_1(z)+2\phi_2(z)\tau]U_\tau+\frac{g(z)}{2}U \tag{B8}$$

instead Eq. (12). The final result of such a treatment is the same as we obtained in Sec. II, and as an example the functions $\phi_n(z)$ for $n=0,1,2$ in this case are

$$\phi_0(z)=a(z)+\tau_c^2c(z),\quad \phi_1(z)=-2\tau_c c(z),\quad \phi_2(z)=c(z). \tag{B9}$$

Thus we have proved that for the class of polynomial form of the phase given by Eq. (B3), only the quadratic case (with $N=2$) or equivalently the phase (B1) for $n=2$ is compatible with the our autonomous principle.

It is important to understand that the autonomous requirement is not equivalent to the quadratic phase requirement, since the quadratic phase given by Eq. (8) does not lead to Eqs. (10) and (11).

The autonomous principle is important because the classes of autonomous and nonautonomous solutions give the proper classification of exact solutions of the generalized NLSE with distributed coefficients. We note, for example, that in the case $\gamma(z)=0$ the exact solutions with quadratic phase are not autonomous.

APPENDIX C: BOUNDED JACOBIAN ELLIPTIC FUNCTIONS

We present in this section the equations for bounded Jacobian elliptic functions in the same form as Eq. (31). Though Eq. (31) may be integrated directly, to avoid complicated transformations one can use the differential equations.

(i) $\tilde{F}(u)=A\ \mathrm{sn}(u,k)$:

$$\left(\frac{\mathrm{d}\tilde{F}}{\mathrm{d}u}\right)^2=A^2-(1+\mathrm{k}^2)\tilde{F}^2+A^{-2}\mathrm{k}^2\tilde{F}^4. \tag{C1}$$

(ii) $\tilde{F}(u)=A\ \mathrm{cn}(u,\mathrm{k})$:

$$\left(\frac{\mathrm{d}\tilde{F}}{\mathrm{d}u}\right)^2=A^2(1-\mathrm{k}^2)+(2\mathrm{k}^2-1)\tilde{F}^2-A^{-2}\mathrm{k}^2\tilde{F}^4. \tag{C2}$$

(iii) $\tilde{F}(u)=A\ \mathrm{dn}(u,\mathrm{k})$:

$$\left(\frac{\mathrm{d}\tilde{F}}{\mathrm{d}u}\right)^2=-A^2(1-\mathrm{k}^2)+(2-\mathrm{k}^2)\tilde{F}^2-A^{-2}\tilde{F}^4. \tag{C3}$$

(iv) $\tilde{F}(u)=A\ \mathrm{sd}(u,\mathrm{k})$:

$$\left(\frac{\mathrm{d}\tilde{F}}{\mathrm{d}u}\right)^2=A^2+(2\mathrm{k}^2-1)\tilde{F}^2-A^{-2}\mathrm{k}^2(1-\mathrm{k}^2)\tilde{F}^4. \tag{C4}$$

(v) $\tilde{F}(u)=A\ \mathrm{cd}(u,\mathrm{k})$:

$$\left(\frac{\mathrm{d}\tilde{F}}{\mathrm{d}u}\right)^2=A^2-(1+\mathrm{k}^2)\tilde{F}^2+A^{-2}\mathrm{k}^2\tilde{F}^4. \tag{C5}$$

(vi) $\tilde{F}(u)=A\ \mathrm{nd}(u,\mathrm{k})$:

$$\left(\frac{\mathrm{d}\tilde{F}}{\mathrm{d}u}\right)^2=-A^2+(2-\mathrm{k}^2)\tilde{F}^2-A^{-2}(1-\mathrm{k}^2)\tilde{F}^4. \tag{C6}$$

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